Some philosphical thoughts concerning Infinitesimal quantities and the infinite

by Arté

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APPROXIMATION OF FRACTIONS BY REPEATED HALVING

On Isla they do not have the concept of fractions as such but undertake division by repeated halving, and in practice rarely go beyond four iterations (division by sixteen).

Thus for most Islanders 1/3 is somewhere beween 5/16 and 3/8. Where greater accuracy is required, there are 'secret' methods for division by 3 and 5 via specially constructed tables, held within the confines of particular religious orders.

When I was still on Isla, I realised that there was no need to be limited to four levels of halving. A fraction is then approximated by a number of iterations with each iteration being an accumulation of halvings, expressed as a double inequality sandwiching the fraction between two fractions with denominators powers of two. For example for 1/3 with 8 iterations yields:

$$0<\frac{1}{3}<\frac{1}{2}\rightarrow\frac{1}{4}<\frac{1}{3}<\frac{2}{4}\rightarrow\frac{2}{8}<\frac{1}{3}<\frac{3}{8}\rightarrow\frac{5}{16}<\frac{1}{3}<\frac{6}{16}\rightarrow\frac{10}{32}<\frac{1}{3}<\frac{11}{32}\rightarrow\frac{21}{64}<\frac{1}{3}<\frac{22}{64}\rightarrow\frac{42}{128}<\frac{1}{3}<\frac{43}{128}\rightarrow$$

$$\frac{85}{256}<\frac{1}{3}<\frac{86}{126}\rightarrow\frac{170}{512}<\frac{1}{3}<\frac{171}{512}$$

giving the final result that 1/3 is between 85/256 and 171/512. The mid value, in this case 341/1024 becomes the 'best' estimate for 1/3.

Once I learned the place value system of representing numbers I realised this amounts to 'decimal' approximations in base 2. In general $c_r/2^r$ is an approximator to the fraction a/b, where,

$c_r = a_0\, 2^r + a_1 2^{r-1} + ... + a_i 2^{r-i} + ... + a_{r-1} 2 + a_r$, with the a_i all being 0 or 1, where

$$\frac{a}{b}-\frac{c_r}{2^r}\leq\frac{1}{2^{r+1}}<\frac{c_r+1}{2^r}-\frac{a}{b}$$

MORE GENERAL APPROXIMATION OF FRACTIONS AND UNFRACTIONS

Here in The Western Federation, there is nothing new about this. Indeed what we do when expressing a fraction as a decimal is exactly the above except with ten replacing two in the denominators and any of the digits 0 to 9 occuring as coefficients of the powers of 1/10. The important point it tells us is that it is possible to establish sequences of fractions that sandwich a given number getting closer and closer to the original number, i.e. **infinitesimally** close.

If all numbers could be written as fractions, this insight would tell us nothing, but we know that the square root of 2 cannot be written as a fraction, but from geometry we know we can construct a line of that length from a unit length, being the hypotenuse of a right-angled triangle with the other two sides both being of unit length. Now, the point is that for unfractional numbers like the square root of two, we can approach them infinitesimally closely by sequences of fractions as

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above. Later I'm going to show that far from being atypical the unfractional numbers in a sense outnumber the fractions, even though we only have a very few examples of actual unfractions.[2]

My contention is that any number that can be represented as a point on a line can be approximated to any degree of accuracy required by means of a sequence of fractions approaching it from above or below (that is with fractions all greater or all less that the number concerned). My intention here is to use this fact to construct any such '**line number'** from fractions alone. This is philosophically important because, while the counting numbers can be defined by counting, zero and the negative numbers by subtraction of counting numbers, and the fractional numbers by division by a counting number, we have no actual construction of the line numbers, other than by geometrical construction, which only covers a few cases.

Suppose s_1, s_2,... and b_1, b_2,... are to sequences of fractional numbers satisfying the three properties

1. The sequence {s} gets bigger and the sequence {b} gets smaller as the sequence goes on, i.e $s_i \leq s_j$ and $b_i \geq b_j$ whenever i < j.

2. All the numbers in the b sequence are greater than all the numbers in the s sequence, i.e. $s_i < b_j$ for all values of i and j.

3. The two sequences approach each other arbitrarily closely, i.e. suppose e is a small fraction > 0 (for example I might use a large power of 1/10), then there is a number n for which $b_i - s_j < e$ for all values of i and j greater than n.

Then I say that these sequences of fractions define a unique line number, namely the point on the line which each approaches as n increases.

The sequences obtained by writing a number to an increasing number of decimal places are examples of sequences of fractions that satisfy these three properties. The sequence {s} is the sequence of increasing numbers of places in the decimal representation of a number, the sequence b is the same sequence but with the latest decimal place increased by 1. For example the sequences, {s} = {1, 1.7, 1.73, 1.732, 1.732, 1.73205,...} and, {b} ={ 2, 1.8, 1.74, 1.733, 1,7321, 1.73206,.. } are the first few terms of sequences defining the square root of 3.

We know that in the case of the decimal expansion that if the expansion ends in a (finite) recurring sequence then the line number represented is fractional and vice versa.(For example 1/3 = 0.3 recurring and 1/6 = 0.16 with the 6 recurring and 1/7 = 0.142587 all recurring). If the expansion continues without such a recurring tail the number must be unfractional.

Although in practical applications we tend to use these decimal approximations, any sequences of fractions satisfying the three properties can be used to approximate a number.

In the above I have confined my attention to approximating a number by sandwiching it beween two sequences. It is possible to modify the properties to achieve the same result using just one sequence that can arbitrarily jump about between values below and above the number being approximated. As before let {s} be a sequence of fractions. Two properties are now needed:

1. Eventually the gaps between successive numbers in the sequence get smaller, i.e. there is a number n such that if i, j > n with i > j then the difference between s_{i+1} and $s_i \leq$ the difference

2 For example those involving the square root of a whole number that is not a perfect square.

between s_{j+1} and s_j (defining difference between a and b to be whichever of a-b or b-a is non-negative).

2. The differences between successive terms approach zero infinitesimally closely, i.e. given any arbitrary positive fraction, e there is a number n such that $-e < s_{i+1} - s_i < e$, for all values of $i > n$.

CONTEMPLATING THE INFINITE

There is an ancient paradox involving a race between a horse and donkey. The horse runs twice as fast as the donkey, which is therefore given a start of, say, one unit. After a while the horse reaches the donkey's starting position, but the donkey is still ahead by half a unit. By the time the horse reaches that point the donkey has moved on and is still ahead, this goes on. How does the horse ever catch the donkey? The point is, of course, that both donkey and horse are moving on a continuous line and approximating by small steps is inappropriate. This got me thinking about different approaches to the infinite.

First there is the infinite like the infinite number of counting numbers. This is weird enough. Because we can leave one out and still have an infinite number left. We can even leave half of them out and stay with the same infinite number. For example we can count the even numbers using all the counting numbers, even though the even numbers themselves are just half of all the numbers. We can count them thus:

Even Numbers: 2, 4, 6, 8 ,10,12,.......

Counting them: 1, 2, 3, 4, 5, 6,.........

We can even count all pairs of numbers, to see this consider setting them out in a square array, as illustrated here:

	1	2	3	4	5	......
1	1, 1	1, 2	1, 3	1, 4	1, 5	.....
2	2, 1	2, 2	2, 3	2, 4	2, 5	.....
3	3, 1	3, 2	3, 3	3, 4	3, 5	......
4	4, 1	4 2	4, 3	4, 4	4, 5.	......
5.	5, 1	5, 2	5, 3	5, 4	5, 5	......

Now starting in the top left corner we can count the pairs as:

Counting:	1	2	3	4	5	6	7	8	9	10	11	12
Pairs:	1, 1	1, 2	2, 1	3, 1	2, 2	1, 3	1, 4	2, 3	3, 2	4, 1	5, 1	4, 2 etc.

This way we see that we can count the pairs of numbers using the counting numbers without missing any out or running out of numbers to count with. Since fractions are just pairs of numbers this means we can also count the fractional numbers, even though we can't list them in order of size.

This is the prelude to my next philosophical discovery. You can't count the line numbers, no matter how you try you will always miss some out. I demonstrate this as follows. To make things easy I'll just try to count the line numbers between 0 and 1. Suppose I can do so, then I can list them in some order 1, 2, 3 etc.

I'm now going to construct a line number between 0 and 1 that is not in the list. Before the decimal point the number will be 0.

After the decimal point I proceed as follows:

Take the first decimal place of the first number in the list, it will contain a number between 0 and 9, a_1 , say. I am going to chose a different value, for technical reasons I will avoid choosing 0 or 9. To be definite, I will choose $b_1 = a_1 +1$ unless a_1 is 8 or 9 when I will choose $b_1 = 1$. This choice will then be the first value after the decimal point in my constructed number.

Do the same with the second number on the list, this time taking the second decimal place, changing it as above to get b_2.

Continue in the same way, fom the nth decimal place of the nth number in the list choose b_n to be different.

Then I will end up with a number $0.b_1b_2b_3...b_n....$ which is a number between 0 and 1. It is also not in my original list because it differs from each number in the list, at least in one decimal place.

This is impossible if I had indeed counted all the line numbers between 0 and 1. That means that the infinity of the line numbers is bigger than the infinity of the counting numbers. But the infinity of the fractional numbers is the same as the infinity of the counting numbers. This means there is a bigger infinity of unfractions than of fractions.

This seems wierd because a different argument can be used to show that between any two unfractions there is at least one fraction (in fact there will be an infinite number of them).

To see this consider two different numbers (fractions or unfractions) $a < b$.

Let $b-a = c > 0$. Now take fractional approximations $a_f \geq a$ and $b_f \leq b$ respectively that are closer than c/3. Then $(b_f - a_f)/2$ is a fractional number between a and b. Indeed the line segment between these fractional numbers is a fractional number greater than c/3 which is non-zero and will thus contain infinitely many other fractional numbers. I said it was weird.

What this means is that when dealing with infinity, care needs to be taken on whether we are dealing with continuous processes like, for example, the asymptotic behaviour of a hyberbola, or discrete processes like counting.